Peter the Great St. Petersburg Polytechnic University

Faculty of Applied Mathematics

The discipline “ Numerical methods”

Explanatory note to educational practice

**The Solution of Numerical Algebraic and Transcendental Equations. Bisection Method and The Method of False Position**

Student: Smirnova D. S.

Group: 3630102/80001

Head of practice : Anufriev I.V.

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Head of practice \_\_\_\_\_\_\_\_\_\_\_\_

Student \_\_\_\_\_\_\_\_\_\_\_\_

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1. Statement of the problem and its formalization

Consider the equations:

We will find all solutions of these equations , using bisection and false position methods, with precision :. For example,

A nonlinear function (𝑥) in its domain of definition 𝐷⊆ℝ may have a finite or infinite number of zeros or none at all. For application of some methods of finding of zeros it is required to know intervals in which obviously there is a root, besides the unique. In other words, there is a subtask of existence and uniqueness, finding boundaries and localization of roots.Алгоритм метода и условия его применимости

* 1. Bisection Method

Conditions of applicability:

1. , где

Algorithm:

A system of nested intervals is created such that . When the length of the interval will stop to exceed the taken accuracy of epsilon, the solution of is any number from the interval. That is:

Rate of convergence:

The middle of the n-th segment is a point gives an approximation to the desired value of , that has an error estimate:

From this estimate, it can be seen that the bisection method converges at a geometric progression rate with the denominator . This is slower compared to other methods, however the method is simple and undemanding.

* 1. The Method of False Position

It is also known as the secant method. Is a modification of the method of dichotomy (or bisection), its advantage is that we take into account the values at the ends of the segment, choose the point c in proportion to these values. That is, it makes sense to find this point as the point of intersection of the abscissa axis with the line connecting

Conditions of applicability:

1. Efficiency is observed at a small length
2. , где

Algorithm:

The approximation will be found from the similarity of triangles

.

Next replace the left border () if and right – in the opposite case. That is, the boundary of the root isolation segment [a, b] for which the sign of the function coincides with the sign of the second derivative is fixed.

The length of the root localization interval in this case may not tend to zero, so usually the count is kept until the values of C coincide on two adjacent iterations with an accuracy of ε, or rather until

≤ . где .

1. Preliminary analysis of the problem (is there a solution and how much)
   1. Study of the equation

Consider . From the construction of the polynomial by means of the MATLAB package, it can be seen that out of 4 possible real roots, the polynomial has all 4 real ones. Two negative and two positive.

The graph of the transcendental equation crosses the abscissa axis 6 times, where one of the six roots is negative. Moreover, it follows from the positivity condition of the logarithm that the function makes sense only when

* 1. Root Separation

Consider . Let us use the theorem on the upper bound of the positive roots of the polynomial:

The first negative coefficient has number 2, so all roots are smaller

. We apply the same theorem to and we get an estimate for the lower roots of the polynomial So the roots are in the interval . Let's try to find more accurate and convenient intervals and separate the roots:

Consider the table:

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| *x* | -9 | -6 | | -3 | -1 | 1 | 2 | 4 | 5.7 |
| *P( x )* | -524.5 | -610 | | -164.5 | -12.5 | -4.5 | -2 | 210 | 913.18 |
|  | decline | | grow | | | | | | |

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| -11 | -7 | -5 | -2 | | -0.5 | 0.1 | | 0.5 | | 1.5 | | 3 |
| 667.5 | -708.5 | -460.5 | -66 | | -1.22 | 1.89 | | -0.22 | | -6.72 | | 51.5 |
| ROOT | |  | | ROOT | | | ROOT | |  | | ROOT | |

So we separated the roots. Consider the third root, lying from 0.1 to 0.5. Find a stationary point for the chord method. Stationary is the point whose value coincides in sign with the value of the second derivative . on the segment from 0.1 to 0.5 is negative. The stationary point will be 0.5.

Consider . Find the intersection of functions and . From their graphs it can be seen that there are only 2 intersections, in the neighborhood of 0, one positive and the other negative. The first function is a parabola, branches up, the minimum is reached at the point *x* = 0. The second function exists on the entire real line, . So, at negative *x*, the function increases, crosses the parabola above the abscissa axis. So

. The maximum at negative is reached at (increasing function) that is it is necessary to solve the inequality . There are about The first time the functions intersect on a segment . The second time пересекает crosses the parabola below zero, then the root is enclosed in the segment that is, on the segment . Let's look for the first root, specify the segment.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *х* | -2 | -1.6 | -1.3 | -1 |
| *F(x)* | 0.87 | 0.079 | -0.42 | -0.85 |

The stationary point for the method of chords:

positive on the interval Hence, the stationary point -1.6.

1. Checking the conditions of applicability of the method
   1. Checking the conditions of applicability of the bisection method

*P(x):* at the ends of the interval the function takes the values of different characters, the root is separated. The derivative is negative on this segment (the sign-constant condition is also satisfied). Sufficient conditions for the bisection method have done.

*F(x)*: at the ends of the interval the function takes the values of different characters, the root is separated. The first derivative is negative on this segment. Sufficient conditions have done.

* 1. Checking the conditions of applicability of the method of false position

The first and second derivatives are continuous on the selected segment. At the ends of the segment, the function takes the values of different characters. The root is separated, in this segment it is unique. About the sign of the first derivative was already mentioned in the previous paragraph, the sign of the second derivative was considered when searching for a stationary point. Sufficient conditions are met.

1. Text example with detailed calculations for a small dimension problem “ hand calculation”)
   1. Calculation for the bisection method

Consider and find the root of polynomial with precision . powers with real coefficients, so it has a maximum of 2 real roots. From the graph (and in this case just from the experience of solving quadratic equations) it is clear that there are 1 positive and one negative real roots. Applying the upper bound theorem we obtain an estimate: , Let's clarify the boundaries.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *x* | -1.58 | -1.2 | -0.7 | -0.3 | 0 |
| *P(x)* | 2.163 | 1.1067 | 0.1567 | -0.2433 | -0.3333 |

So, I will look for the root in the interval from -0.7 to -0.3. sign is constant on this interval. is positive on the entire number line. All sufficient conditions are met, we will carry out the calculation

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Итерация | *а* | *b* |  | *x\** |  |
| 1 | -0.7 | -0.3 | -0.5 | -0.577350269189626 | -0.0833333 |
| 2 | -0.7 | -0.5 | -0.6 | 0.0266667 |
| 3 | -0.6 | -0.5 | -0.55 | -0.0308333 |
| …. | | | | | |
| 47 | -0.5773502691896284 | -0.5773502691896227 | -0.5773502691896255 | -0.577350269189626 | -2.775557561562891e-16 |
| 48 | -0.5773502691896284 | -0.5773502691896255 | -0.5773502691896269 | 1.38777878078144e-15 |

In 48 iterations, we got the root with an accuracy of 15 decimal places.

Take the equation and find its root at with precision . straight. A decreasing function passing through the origin. the logarithmic function that has meaning on the interval , increases. The functions intersect in 4 quarters, so that one can choose for the initial consideration the interval from 0 to such that . That is the segment . Let us clarify the considered interval.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *x* | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
| *F(x)* | -2.2 | -0.9 | -0.2 | 0.3 | 0.8 |

So, I will look for the root on the interval from 0.5 to 0.7. at the selected interval(!) continuous, positive, decreasing. at the selected interval(!) is continuous, negative, increasing. Sufficient conditions have done, we will carry out the calculation:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Итерация | *а* | *b* |  | *x\** |  |
| 1 | 0.5 | 0.7 | 0.6 | 0.5671432904097838 | 0. 089174376234009256 |
| 2 | 0.5 | 0.6 | 0.55 | -0.047837000755620362 |
| …. | | | | | |
| 46 | 0.56714329040977984 | 0.56714329040978551 | 0.56714329040978262 | 0.5671432904097838 | -3.4416913763379853e-15 |
| 47 | 0.56714329040978262 | 0.56714329040978551 | 0.56714329040978406 | 5.5511151231257827e-16 |

* 1. Calculation for the method of false position

Polynomial:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Итерация | *а* | *b* |  | *x\** |  |
| 1 | -0.7 | -0.3 | -0.54333333333333333 | -0.577350269189626 | -0.038122222222222224 |
| 2 | -0.7 | -0. 543333333333333 | -0.57399463806970508 | -0.003863488800561587 |
| …. | | | | | |
| 14 | -0.7 | -0.5773502691896041 | -0.57735026918962362 | -0.577350269189626 | -2.442490654175344e-15 |
| 15 | -0.7 | -0.5773502691896236 | -0.57735026918962551 | -2.775557561562891e-16 |

Transcendental:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Итерация | *а* | *b* |  | *x\** |  |
| 1 | 0.5 | 0.7 | 0.57200640308114237 | 0.5671432904097838 | 0.013401309613950030 |
| 2 | 0.5 | 0.57200640308114237 | 0.56733447301249429 | 0.00052822332873314615 |
| …. | | | | | | |
| 10 | 0.5 | 0.56714329040981215 | 0.56714329040978495 | 0.5671432904097838 | 2.9976021664879227e-15 |
| 11 | 0.5 | 0.56714329040978495 | 0.56714329040978384 | -1.110223024625156e-16 |

1. Preparation of control tests for method illustration by means of MATLAB package

The tests are prepared as follows: take the equations in question at selected intervals and consistently analyze the dependence of the number of iterations of the method on the initial approximation of the point and the dependence of the number of iterations on the required accuracy of the root.

Total, constant in the program will be the following values:

For polynomial : the boundaries of the considered segment are the left boundary а = 0.1, the right boundary b = 0.5, the numerical answer with which we will compare obtained by means of MATLAB x\* ≈ 0.471307533879952.

For : the left boundary a = -1.6, the right boundary b = -1.3, the numerical answer \* ≈ -1.5553465556979733500.

The accuracy of in both cases will vary from 0.1 to 1.е-15 (). The initial approximation will vary within the selected segments (until we approach the root at a distance of 4 ε). Constants will be declared as #define.

On the basis of the obtained results, we will build dependency graphs in MATLAB.

1. Modular structure of the program
2. Algorithms of methods are implemented in C language. Input parameters: the boundaries of the considered segment, the investigated function, a pointer to the number of iterations (counter), the accuracy of the calculation of Epsilon. Output parameter: changed number of iterations lying on the passed pointer, numerical response.
3. Processing of the results and writing them to a structured file is carried out by a special function Algorithm. It takes the input stream, which will output information and the necessary values for calculations (constants described earlier).
4. The Received files with values are opened in MATLAB and on the basis of these data dependency graphs are built.
5. Numerical analysis of the problem solution

In the process of implementing the decision was made in the method of the chords also to consider the value of the function is the newly found approximate values to compare signs point values to replace a or b. After the answer is calculated, it is returned in the function Algorithm, which was previously created arrays, where is written: length of the considered cut (approximation) and the number of iterations and the accuracy Epsilon and the number of iterations – this is done sequentially, separated by a space. The function also checks that the condition is true. If everything is successful-the standard output stream sends a message "SUCCESSFUL".

Let's consider the third step of the analysis of the methods of the previous paragraph in more detail. We understand the dependencies in separate graphs, consider in pairs: in the half-division method, the dependence of the number of iterations on the initial approximation decreases more slowly as the segment narrows than in the chord method (by 11.5% in the first method and by 42.8% in the second), The same pattern is observed with the refinement of the root (reduction of the Epsilon error). The rate of convergence (number of iterations) as a function of the approximation and of the Epsilon repeats for the Transcendental function as well, so that the ratio of methods is the same..

1. Summary
   1. Comparison of the convergence rate of algorithms

From the numerical analysis of the solution of the problem, it follows that the secant method converges much faster than the half-division method (and with the "improvement" of the conditions, the speed also grows faster). Let's not forget that we have chosen relatively small areas on the number line and in some cases the chord method can converge even slower than the half division method. It even happens that much slower.

* 1. Behavior of unsuitable functions under sufficient conditions

First let's look at the conditions of the bisection method

– f(a)\*f(b) < 0 and 2 roots at the interval

Let's try to find the roots on the segment containing 2 roots, that is, on . In this case, the method converged, but only 1 root was found. A condition was also violated: f(a)\*f(b) < 0. The method converged because at the first iteration f(c) became greater than 0, and f(a) was negative. After that, the algorithm continued to consider the interval and the method converged.

– the continuity of the function

Consider the function , having a gap at point 0 for example at the interval. First iteration – the segment narrows to and in the next iteration we find the root. The method also converged under a broken condition. But it could happen that we would divide by 0 or get a break point of the function instead of an answer.

Let's try to violate sufficient conditions for the chord method. Let's repeat the previous cases.

- 2 roots and the second derivative changes sign.

Roots of at the interval. In this case, the sign of the second derivative is not constant. The method converges! In the first step the chord crosses the x axis at a point . But after the temporary result is calculated as

and then the calculations approach the root.

- the continuity of the function

Consider the function , having a gap at point 0 for example at the interval. iteration – the segment narrows to . Next, at each iteration, we change the beginning and bring it closer to the -1 – root of the equation in this segment. The method worked.

So, one has to be very careful breaking sufficient convergence conditions. But it is still possible to achieve convergence in some cases described above, for example.